Critical Slenderness of Compression Members With Effective Lengths About Nonprincipal Axes

by
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Introduction

For most compression members, the principal axes are oriented such that the designer can evaluate the effective length of the member about each of the principal axes. For an axially loaded member it is a simple matter of comparing the larger of k_xL/r_x and k_yL/r_y to determine the axial capacity of the member. When column moments exist, the stress ratios can be evaluated about both the x and y–axes using the appropriate combined stress equations to determine the critical condition.

However compression members such as single angle struts are typically positioned in structures such that their x and/or y–axes are oriented parallel to the framing. Thus effective lengths can readily be evaluated (or estimated) only about these x and y–axes which are non–principal axes. The minimum radius of gyration is at an angle to the framing and the effective length values.

This does not present a problem if the effective length about the x and y–axes are the same, because this effective length can be used with the z–axis radius of gyration to obtain the largest slenderness. However, if the effective length factors k_x and k_y differ, there is no accepted means of determining the critical slenderness ratio. As a result, the designer typically ignores the end restraint and conservatively uses the actual length and the z–axis radius of gyration to determine a design slenderness ratio.

To take advantage of end restraint in compression members such as angles, a procedure is developed to evaluate an effective minimum radius of gyration based on the x and y–axis effective length factors. The procedure is general and thus is applicable not only to angles, but to any compression member which is oriented such that the effective length factors can not be directly evaluated about the principal axes.

Development

The Euler load can be written as

$$P_e = \frac{\pi^2 EI}{(kL)^2} = \frac{\pi^2 E\left(I/k^2\right)}{L^2} \tag{1}$$

which shows that I/k^2 can be considered as an effective moment of inertia for a column with a buckling length of L. It is presumed at this point that the critical load can be satisfactorily determined from the Euler equation rather than from the flexural–torsional buckling expression.

This effective moment of inertia can be evaluated from the basic integral expressions for the moments of inertia by replacing the actual distance from the neutral axis by this distance divided by the appropriate effective length factor. Thus

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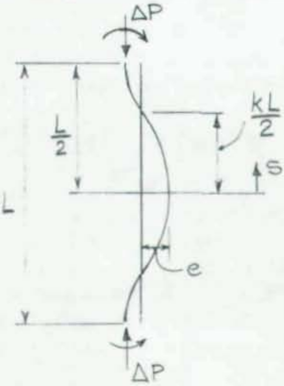
$$I_{x_{eff}} = \int (y/k_x)^2 dA = \frac{I_x}{k_x^2}$$

$$I_{y_{eff}} = \int (x/k_y)^2 dA = \frac{I_y}{k_y^2} \qquad (2)$$

$$I_{xy_{eff}} = \int \frac{x}{k_y}\frac{y}{k_x} dA = \frac{I_{xy}}{k_x k_y}$$

This is akin to considering the cross–section as having orthotropic properties.

The lateral stiffness of a compression member, with initial deformations corresponding to the buckled shape, can be shown to correspond to that indicated by Equation (1). This represents further proof that the lateral bending stiffness should be used for buckling as well.



The equation of the curve is e cos $\pi s/kL$ which means that a differential load ΔP introduces a moment ΔP e cos$\pi s/kL$. By integration it can be determined that the lateral deflection δ is

$$\delta = \frac{\Delta P \cdot e \cdot (kL)^2}{\pi^2 EI} \cos\frac{\pi s}{kL} = \frac{\Delta P \cdot e \cdot L^2}{\pi^2 E\left(\frac{I}{k^2}\right)} \cos\frac{\pi s}{kL}$$

The lateral stiffness is proportional to I/k^2. Thus use of I/k^2 as an effective moment of inertia for stability purposes appears to be appropriate.

The equation for the minimum principal moment of inertia of a cross section given I_x, I_y and I_{xy} is

$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}. \qquad (3)$$

Thus if the x and y properties are represented by the effective values given in Equations (2) then

$$\text{Effective } I_{min} = \frac{I_x}{2k_x^2} + \frac{I_y}{2k_y^2} - \sqrt{\left(\frac{I_x}{2k_x^2} - \frac{I_y}{2k_y^2}\right)^2 + \left(\frac{I_{xy}}{k_x k_y}\right)^2} \qquad (4)$$

A plot of Eq. (4) is illustrated in Fig. 1.

r_{eff}, the corresponding effective minimum radius of gyration can be obtained from the square root of the above expression divided by the area A.

$$r_{eff} = \sqrt{\tfrac{1}{2}\left[\left(\frac{r_x}{k_x}\right)^2 + \left(\frac{r_y}{k_y}\right)^2\right] - \sqrt{\tfrac{1}{4}\left[\left(\frac{r_x}{k_x}\right)^2 - \left(\frac{r_y}{k_y}\right)^2\right]^2 + \left(\frac{I_{xy}}{Ak_xk_y}\right)^2}} \qquad (5)$$

This means that the effective slenderness is L/r_{eff} where L is the unbraced length. When $I_{xy}=0$ such that x and y are principal axes, r_{eff} is the minimum of r_x/k_x or r_y/k_y as would be expected. The author in an earlier presentation proposed consideration of an effective radius of gyration (without proof) that was similar to Equation 5[1].

Should evaluation of the flexural–torsional buckling load be appropriate, the r_{eff} above can be used in computing the value of F_{ez} in the flexural–torsional expression (Eq. C4–2 in Ref. 2). The value of r_{eff} would replace r_z/K_z. A maximum effective radius of gyration, obtained by using a plus sign for the inner square root term in Eq. 5, would replace r_w/K_w in the expression for F_{ew}. The coordinates of the shear center z_o and w_o with respect to the centroid would be replaced by coordinates consistent with the orientation of the effective radii of gyration.

Trahair[3] in 1969 did examine single angles restrained about arbitrary axes. His work was based on a theoretical development using the differential equilibrium equations for major and minor axis bending and for torsion, and incorporating elastic end restraining moments. The equations consider the flexural–torsional behavior of struts, but the solution is difficult to obtain and thus the procedure does not lend itself to design usage.

Evaluation of I_{xy}

I_{xy} in Eqs. 4 and 5 can be determined from basic principals. However, it is often possible to evaluate I_{xy} from other section properties. Having I_z, the minimum principal moment of inertia, one can determine

$$I_{xy} = \sqrt{\left(\frac{I_x + I_y}{2} - I_z\right)^2 - \left(\frac{I_x - I_y}{2}\right)^2} \qquad (6)$$

which reduces to $I_{xy} = I_x - I_z$ when $I_x = I_y$. For angles I_z would be evaluated as Ar_z^2. Using the tabulated tan α given for unequal leg angles, I_{xy} can be evaluated as

$$I_{xy} = \left(I_x - I_y\right)\tan\alpha / \left(1 - \tan^2\alpha\right) \qquad (7)$$

Alternatively for angles, a good value if $I_{xy}/A = r_{xy}^2$ can be determined from

$$I_{xy} / A = \left[\frac{(b - t/2)(d - t/2)}{2(b + d - t)}\right]^2 \qquad (8)$$

where b and d are the leg lengths and t is the thickness of the angle. For equal leg angles one can simply consider $I_{xy}=0.6I_x$

Effective Length Factors for Angle Web Members

For single angles used as web members of a truss, most chords, typically provide significant restraint in the plane of the truss and substantially less restraint about the axis of the chord. The effective length in the plane of the truss could range from 1.0 to 0.65. The out-of-plane effective length is more likely to range from 1 to 0.9. To illustrate the influence of these variations of effective length, the r_x/r_{eff} ratios for several angles are evaluated. The r_x/r_{eff} ratio represents the resultant effective length factor for the angle.

In Fig. 2, the plot illustrates the effect of angles L3x3x1/4, L4x3x1/4 and L5x3x1/4 having the three inch leg welded or bolted (with more than one bolt) to the chord by showing k_y varying from 1.0 down to 0.5. Plots are shown for k_x=1 and 0.9. The plot shows that with a larger leg projecting from the chord, a significant reduction in resultant effective length factor occurs from restraint about the y-axis. This effective length factor approaches the value of k_y as the projecting leg is lengthened as would be expected. Also, as the projected leg becomes larger, the value of k_x has a decreasing influence on the resultant effective length factor.

In Fig. 3, the plot illustrates the effect of the longer leg of the angle attached to the chord. As one would expect, the resultant effective length factor is not altered as much when the significant restraint is about the stronger axis. The resultant effective length factors in this case are influenced more by the value of k_y.

Although the resultant effective length factor plots in Figures 2 and 3 are for a specific set of angles, they can be used for other angles as well. Since the ratio of r_x/r_y is approximately the same for other thicknesses of the angle sizes in the plots, the effective length factors for other thicknesses can be obtained by simply using the plot for the appropriate leg lengths. If the ratio of angle leg lengths is proportional to one of those plotted, the plotted curve can also be used to obtain the effective length factor desired. For example, the 8x6 angle effective length factor can be obtained using the plot for the L4x3x1/4. The resultant effective length factor for any angle can be estimated with good accuracy by interpolation using either the ratio of leg lengths or ratio of r_x/r_y.

Illustrative Examples

For the general situation with compression struts having effective lengths about nonprincipal axes where Figures 2 and 3 could not be used, one would have to obtain the desired slenderness properties directly from Eq. (4) or Eq. (5). The following two examples illustrate the general use of the expressions developed.

Example 1. Determine the allowable axial capacity of an L4x3x5/16 leg of 9′ length. The top is framed into channels which are attached to adjacent structure for bracing, while the bottom consists of a base plate anchored to a concrete foundation.

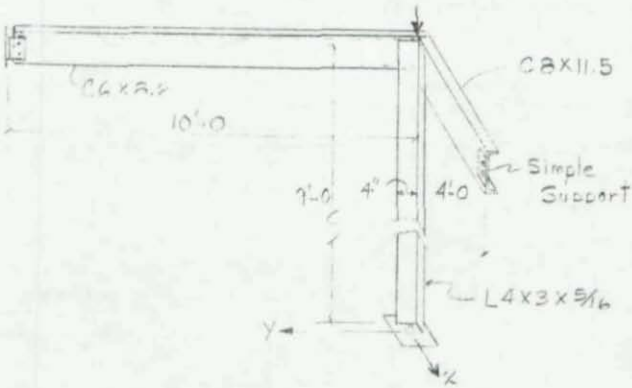
Determine effective length factors from alignment charts.

About x-axis – $G_{top} = \dfrac{3.38/9}{.75(13.1/10)} = 0.38^*$ Use $G_{bot} = 10$ (Connected to foundations)

$\therefore\ k_x = 0.785$ from alignment chart for braced frames.

About y-axis – $G_{top} = \dfrac{1.65/9}{.75(32.6/4)} = 0.030^*$ $G_{bot} = 10$

$$\therefore\ k_y = 0.70$$



$$\frac{r_x}{k_x} = \frac{1.27}{.785} = 1.618 \qquad \frac{r_y}{k_y} = \frac{.887}{.70} = 1.267$$

Using Eq. 7 – $I_{xy} = (3.38 - 1.65)\ 0.554/(1-.554^2) = 1.383$

$I_{xy} / (Ak_xk_y) = 1.383/(2.09 \times .785 \times .70) = 1.204$

From Eq. 5 – $r_{eff} = \sqrt{\frac{1}{2}\left[(1.618)^2 + (1.267)^2\right] - \sqrt{\frac{1}{4}\left[(1.618)^2 - (1.267)^2\right]^2 + (1.204)^2}} = 0.897$

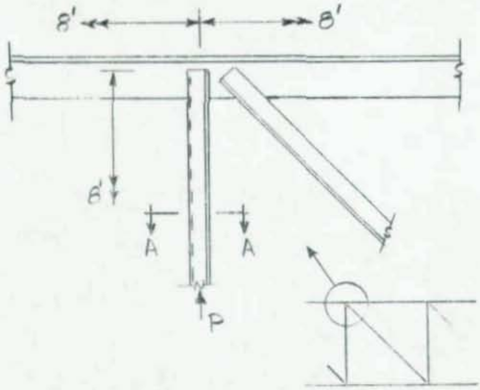
The slenderness ratio is 9(12)/.897 = 120; F_a = 10.28 ksi.

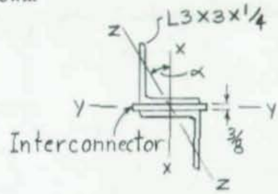
The allowable axial capacity is 10.28(2.09) = 21.48 kips.

*I_x and I_y of the L4x3x5/16, used in computation of G, are larger than the effective moment of inertia for the respective axes and thus a somewhat conservative G is obtained. See Example 2 where computation using a reduced moment of inertia is illustrated.

Example 2

Determine the axial capacity of a 8′ long truss web member in compression. The top and bottom chords are both WT6x25 sections with panels 8′ in length. The member is a double L3x3x1/4 in which the angles are positioned so as to form a Z-shape as shown.





Section A–A

$A = 2.88$ in.^{2}
$I_x = 3.727$ in.^{4}
$I_y = 5.532$ in.^{4}
$I_{xy} = -3.42$ in.^{4}
$\alpha = 37.6°$
$I_z = 1.09$ in.^{4}
$I_w = 8.17$ in.^{4}

Determine effective length factor in plane of truss. First find a reduced stiffness for the double angle in this plane from

$$\frac{1}{I_r} = \frac{\cos^2 37.6}{1.09} + \frac{\sin^2 37.6}{8.17} = 0.621;\ I_r = 1.61$$

since bending is not about a principal axis.

Therefore for both top and bottom

$$G = \frac{1.61/8}{2x18.7/8} = 0.043$$

ignoring the minor benefit of the diagonal.

From the alignment charts find $k_x = 0.522$.

Conservatively consider $k_y = 1.0$

$$\frac{I_x}{2k_x^2} = \frac{3.727}{2(.522)^2} = 6.84 \qquad \frac{I_y}{2k_y^2} = \frac{5.532}{2(1)^2} = 2.766$$

$$\frac{I_{xy}}{k_x k_y} = \frac{-3.42}{(.522)(1)} = -6.55$$

$$\text{Effective } I_{min} = 6.84 + 2.766 - \sqrt{(6.84 - 2.766)^2 + (-6.55)^2}$$

$$= 9.606 - 7.715 = 1.891 \text{ in.}^4$$

$$r_{eff} = \sqrt{1.891/2.88} = 0.810 \text{ in.}$$

whereas $r_z = \sqrt{1.091/2.88} = 0.6155$ in.

which means $k_{eff} = .6155/0.810 = 0.76$

Effective slenderness = 964/0.810 = 118.5

$F_a = 10.5$ ksi for A36 steel.

Allowable capacity P = 2.88(10.5) = 30.24 kips

References

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2. American Institute of Steel Construction, Inc., "Specification for Allowable Stress Design of Single Angle Members – Commentary", Manual of Steel Construction, Allowable Stress Design, Ninth Edition, 1989.

3. Trahair, N.S., "Restrained Elastic Beam–Columns," Journal of the Structural Division, ASCE, Vol. 95, No. ST12, Dec., 1969, pp. 2641–2663.

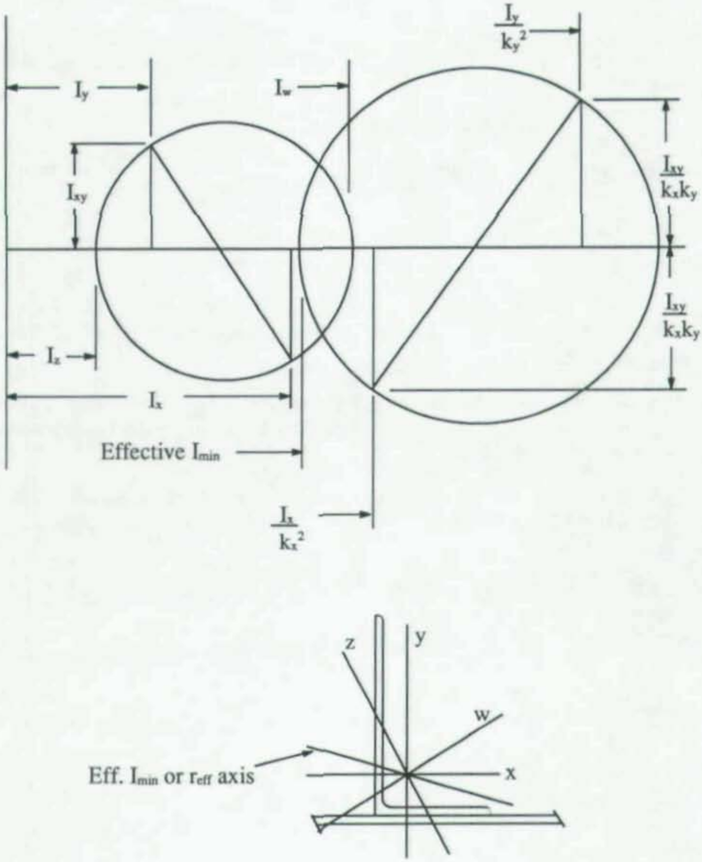

FIGURE 1 - Plot of the Effective Moment of Inertia

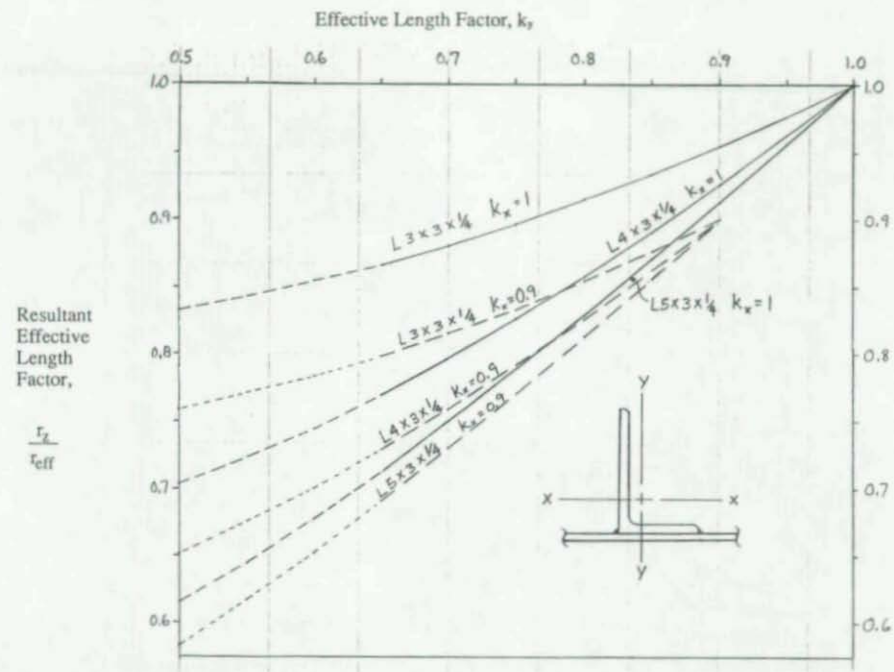

FIGURE 2 – Resultant Effective Length Factor with Primary Restraint About y–axis of Angle

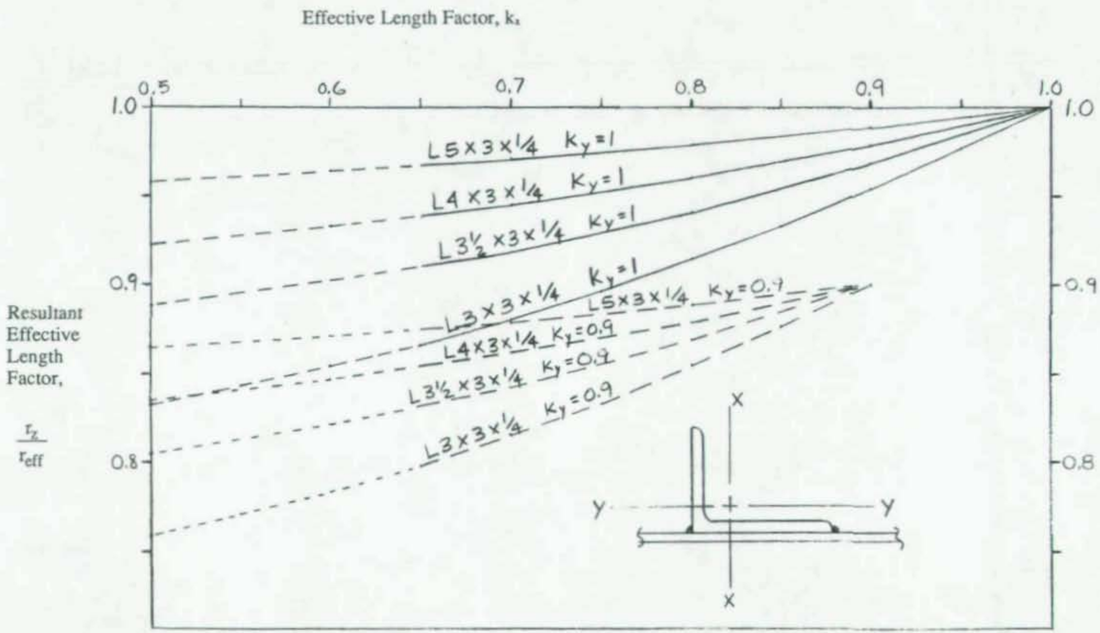

FIGURE 3 – Resultant Effective Length Factor with Primary Restraint About x–axis of Angle